

UNIQUENESS OF BUTSON HADAMARD MATRICES OF SMALL DEGREES

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ABSTRACT. For positive integers m and n , we denote by $\text{BH}(m, n)$ the set of all $H \in M_{n \times n}(\mathbb{C})$ such that $HH^* = nI_n$ and each entry of H is an m -th root of unity where H^* is the adjoint matrix of H and I_n is the identity matrix. For $H_1, H_2 \in \text{BH}(m, n)$ we say that H_1 is *equivalent* to H_2 if $H_1 = PH_2Q$ for some monomial matrices P, Q whose nonzero entries are m -th roots of unity. In this paper we classify $\text{BH}(17, 17)$ up to equivalence by computer search.

1. INTRODUCTION

Following [1], we call an $n \times n$ complex matrix H a *Butson-Hadamard matrix of type (m, n)* if each entry of H is an m -th root of unity and $HH^* = nI_n$ where H^* is the conjugate transpose of H and I_n is the $n \times n$ identity matrix. We denote by $\text{BH}(m, n)$ the set of all Butson-Hadamard matrices of type (m, n) . We give an equivalence relation on $\text{BH}(m, n)$: $H_1, H_2 \in \text{BH}(m, n)$ are *equivalent* if H_2 can be obtained from H_1 via a finite sequence of the following operations:

- (O1) a permutation of the rows (columns);
- (O2) a multiplication of a row (column) by an m -th root of unity.

In this paper we focus on $\text{BH}(p, p)$ where p is a prime. It is well-known that the *Fourier matrix* $F_p = (\exp \frac{2\pi\sqrt{-1}ij}{p})_{0 \leq i, j \leq p-1}$ of degree p is in $\text{BH}(p, p)$ for each prime p , but it is still open whether or not every matrix in $\text{BH}(p, p)$ is equivalent to F_p . On the other hand it would be a quite exciting result if we could find a matrix in $\text{BH}(p, p)$ which is not equivalent to F_p . Because, such a matrix gives rise to a non-Desarguesian projective plane of order p (see Proposition 3.4).

One may get a positive answer for the uniqueness of the equivalence classes on $\text{BH}(p, p)$ for $p = 2, 3, 5, 7$ without any use of computer, and also for $p = 11, 13$ with a light support of computer. (The complexity over 3.0 GHz CPU is about less than 10 seconds.) But, for larger prime numbers p , one may notice that a heavy amount of complexity is needed in order to classify matrices in $\text{BH}(p, p)$. In fact it was estimated to take about 5000 hours in order to do it for $\text{BH}(17, 17)$ over a single 3.0

¹This work was supported by the Financial Supporting Project of Long-term Overseas Dispatch of PNU's Tenure-track Faculty, 2012.

²This work was supported by Kyushu University Friendship Scholarship.

GHz CPU. We introduced a parallel algorithm to solve the following result. The computation is executed on the high performance multi-node server system Fujitsu Primergy CX400 in Kyushu University.

Theorem 1.1. *For a prime $p \leq 17$, every matrix in $\text{BH}(p, p)$ is equivalent to the Fourier matrix of degree p .*

In section 2 we explain our algorithm to find up to equivalence all the matrices in $\text{BH}(p, p)$. In section 3 we will prove that if there is a matrix in $\text{BH}(p, p)$ which is not equivalent to the Fourier matrix F_p then there exists a non-Desarguesian projective plane of order p .

2. ALGORITHM TO CLASSIFY $\text{BH}(p, p)$

Throughout this paper the entries of an $n \times n$ matrix is indexed by integers from 0 to $n - 1$. For instance, the upper leftmost entry is considered to be in $(0, 0)$ -position rather than $(1, 1)$ -position, and lower rightmost entry is in $(n - 1, n - 1)$ -position than (n, n) -position.

In the sequel we assume that p is prime and

$$\xi_p = \cos(2\pi/p) + \sqrt{-1} \sin(2\pi/p).$$

We denote by $\mathbb{F}_p = \{0, 1, \dots, p - 1\}$ a finite field with p elements, and adopt the natural ordering of \mathbb{F}_p , i.e., $0 < 1 < \dots < p - 1$.

Definition 2.1. We say that $D = (D_{i,j}) \in M_{p \times p}(\mathbb{F}_p)$ is a *difference matrix* if $\mathbb{F}_p = \{D_{i,k} - D_{j,k} \mid k = 0, 1, \dots, p - 1\}$ for any i and j with $i \neq j$. The set of all difference matrices of degree p is denoted by $\mathcal{D}(p)$.

We define a map $\lambda : \text{BH}(p, p) \rightarrow M_{p \times p}(\mathbb{F}_p)$ by $\lambda(H) = (E_{i,j})$ for $H = (\xi_p^{E_{i,j}}) \in \text{BH}(p, p)$. (Since $(\xi_p)^p = 1$ we can regard an exponent $E_{i,j}$ as an element of \mathbb{F}_p .)

Lemma 2.2. *The map λ is one to one and $\text{Im } \lambda = \mathcal{D}(p)$. So there is a one to one correspondence between $\text{BH}(p, p)$ and $\mathcal{D}(p)$.*

Proof. The injectivity follows from the definition of λ . Let $H = (\xi_p^{E_{i,j}}) \in \text{BH}(p, p)$. Then, for all distinct i, j with $0 \leq i, j \leq p - 1$,

$$(HH^*)_{i,j} = \sum_{k=0}^{p-1} H_{i,k} \bar{H}_{j,k} = \sum_{k=0}^{p-1} \xi_p^{E_{i,k} - E_{j,k}}.$$

Since $x^{p-1} + \dots + x + 1$ is the minimal polynomial of ξ_p , $(HH^*)_{i,j} = 0$ if and only if $\{E_{i,k} - E_{j,k} \mid k = 0, 1, \dots, p - 1\} = \mathbb{F}_p$. Hence $\lambda(H) \in \mathcal{D}(p)$ and λ is onto $\mathcal{D}(p)$. \square

For $D = (D_{i,j}) \in \mathcal{D}(p)$ we say that D is *fully normalized* if $D_{0,i} = D_{i,0} = 0$ and $D_{1,i} = D_{i,1} = i$ for all $i = 0, 1, \dots, p - 1$. For $H \in \text{BH}(p, p)$, H is called *fully normalized* if so is $\lambda(H)$. If $N = (N_{i,j})$ in $\mathcal{D}(p)$ (in $\text{BH}(p, p)$, respectively) is fully normalized then the $(p - 2) \times (p - 2)$ submatrix $(N_{i,j})_{2 \leq i, j \leq p-1}$ is called the *core* of N .

Classifying $\text{BH}(p, p)$ is equivalent to finding all possible cores of fully normalized matrices in $\text{BH}(p, p)$. For convenience we can move our workspace to $\mathcal{D}(p)$ due to Lemma 2.2. The next proposition shows that there is a systematic way to find a difference matrix:

Proposition 2.3. *Let $L = (L_{i,j}) \in M_{p \times p}(\mathbb{F}_p)$. Then, $L \in \mathcal{D}(p)$ if and only if $L_{i,j} \neq L_{i,b} + L_{a,j} - L_{a,b}$ for all $0 \leq a < i \leq p-1$ and $0 \leq b < j \leq p-1$.*

Proof. (\Rightarrow) By the definition of a difference matrix we have $L_{i,j} - L_{a,j} \neq L_{i,b} - L_{a,b}$. (\Leftarrow) Fix i and a . Then $\{L_{i,k} - L_{a,k} \mid k = 0, \dots, p-1\} = \mathbb{F}_p$ by the condition. \square

Fix i and j with $0 < i, j \leq p-1$. Then Proposition 2.3 tells us that if we hope to determine the (i, j) -entry of a difference matrix then we have to check the condition $L_{i,j} \neq L_{i,b} + L_{a,j} - L_{a,b}$ for all a and b with $0 \leq a < i$ and $0 \leq b < j$. This leads the following algorithm:

Algorithm, $C(i, j)$:

Input: $i, j \in \{1, \dots, p-1\}$ and a $p \times p$ matrix $L = (L_{i,j})$

Output: $r(i, j)$ (a subset of \mathbb{F}_p)

$r(i, j) \leftarrow \mathbb{F}_p$; $a \leftarrow 0$; $b \leftarrow 0$

WHILE $0 \leq a < i$ DO

 WHILE $0 \leq b < j$ DO

$r(i, j) \leftarrow r(i, j) \setminus \{L_{i,b} + L_{a,j} - L_{a,b}\}$

$b \leftarrow b + 1$

$a \leftarrow a + 1$

RETURN $r(i, j)$

The algorithm $C(i, j)$ returns a set $r(i, j)$ of candidates for the entry $L_{i,j}$ if the upper left entries $L_{a,b}$ ($0 \leq a < i$ and $0 \leq b < j$) are already determined.

Now suppose that we hope to construct a fully normalized matrix in $\mathcal{D}(p)$. Let L be a matrix in $M_{p \times p}(\mathbb{F}_p \cup \{\perp\})$ such that

$$(1) \quad L_{0,i} = L_{i,0} = 0, \quad L_{1,i} = L_{i,1} = i \text{ and } L_{j,k} = \perp$$

for all $i \in \{0, \dots, p-1\}$ and $2 \leq j, k \leq p-1$ where $\mathbb{F}_p \cap \{\perp\} = \emptyset$. (The letter ' \perp ' stands for the 'empty' entry.) In the sequel we should fill the core of L by using the algorithm $C(i, j)$ so that $L \in \mathcal{D}(p)$. First of all we need an appropriate order of computation which is compatible to the algorithm $C(i, j)$:

Definition 2.4. Let $\mathcal{I} = \{(i, j) \mid 2 \leq i, j \leq p-1\}$ be the set of indices of the core of L . A total order \preceq on \mathcal{I} is called *admissible* if the following conditions hold.

- (i) For all $(i, j) \in \mathcal{I}$ we have $(2, 2) \preceq (i, j)$;

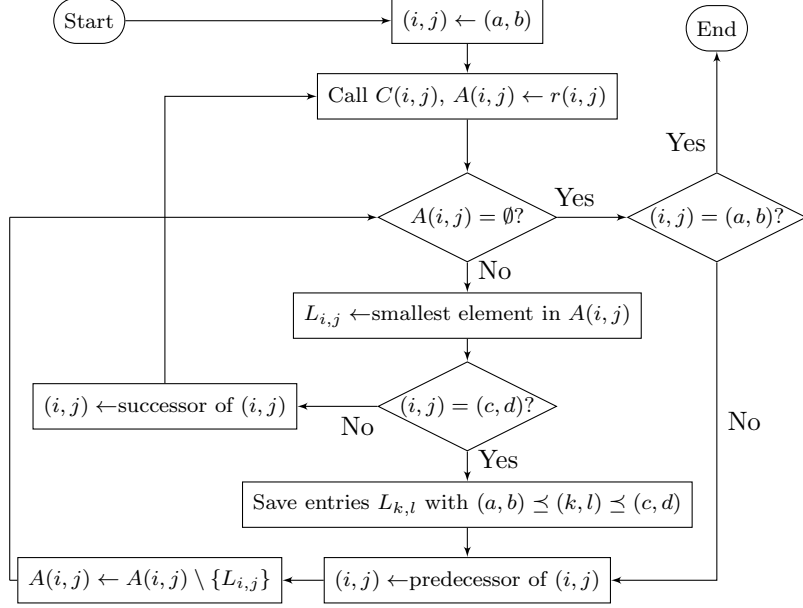


FIGURE 1. The main algorithm, $M(a, b, c, d)$.

- (ii) For any $(i, j) \in \mathcal{I}$, if $2 \leq k \leq i$, $2 \leq l \leq j$ then $(k, l) \preceq (i, j)$.

Example 2.4.1. The following are admissible total orders on \mathcal{I} .

- (i) Diagonal order 1, \preceq_D : $(2, 2) \prec (2, 3) \prec (3, 2) \prec (3, 3) \prec (2, 4) \prec (3, 4) \prec (4, 2) \prec (4, 3) \prec (4, 4) \prec \dots$.
- (ii) Diagonal order 2, $\preceq_{D'}$: $(2, 2) \prec (2, 3) \prec (3, 2) \prec (2, 4) \prec (3, 3) \prec (4, 2) \prec (2, 5) \prec (3, 4) \prec (4, 3) \prec (5, 2) \prec \dots$.
- (iii) Horizontal order, \preceq_H : $(2, 2) \prec (2, 3) \prec \dots \prec (2, p-1) \prec (3, 2) \prec \dots \prec (3, p-1) \prec (4, 2) \prec \dots$.

With an admissible total order \preceq on \mathcal{I} we now introduce the main algorithm $M(a, b, c, d)$. See Figure 1. Notice that the parameter (a, b) (respectively, (c, d)) indicates the starting (resp. finishing) index of the algorithm. For example, by calling $M(2, 2, p-1, p-1)$, we can obtain all possible cores of fully normalized matrices in $\mathcal{D}(p)$.

There is a redundancy in our algorithm. Notice that if there exists a matrix A in $\mathcal{D}(p)$ then the transpose A^T is also in $\mathcal{D}(p)$, because the initial part (cf. the equation (1)) of the construction for L is symmetric. Although A and A^T may not be equivalent it is sufficient to find only one of A and A^T in the searching algorithm, and we just add each transpose to the result in the final step. Therefore we may assume

$$(2) \quad L_{2,3} \leq L_{3,2}.$$

For primes $p \leq 13$ the main algorithm $M(2, 2, p-1, p-1)$ works well. Over 3.0 GHz CPU within less than 10 seconds, we obtain the following result: For a prime $p \leq 13$, there is a unique fully normalized matrix in $\text{BH}(p, p)$, namely, the Fourier matrix of degree p .

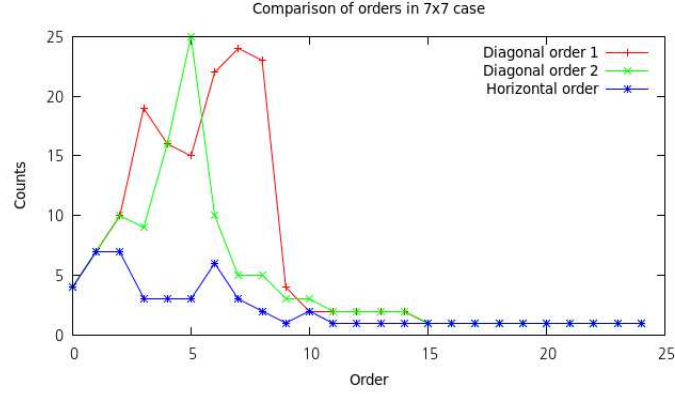


FIGURE 2. The case $p = 7$.

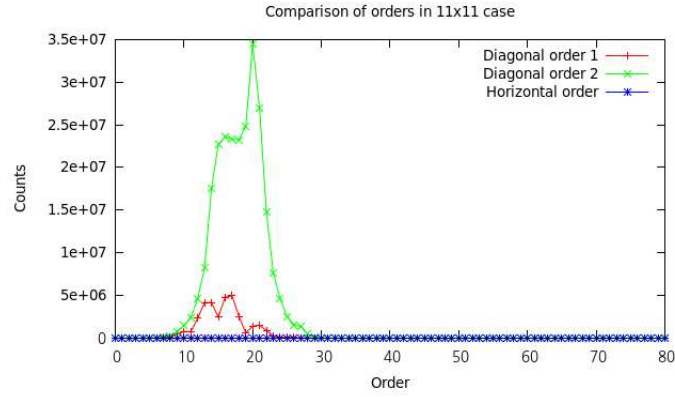


FIGURE 3. The case $p = 11$.

The next case $p = 17$ needs a heavy computer calculation. So we use a parallel algorithm to use a supercomputer. Our strategy is given as follows: Let (r, s) be a fixed index among a total order \preceq . The *master thread* carries out $M(2, 2, r, s)$. If there is a *partial solution* from $(2, 2)$ to (r, s) then the master process passes this partial information of the matrix L to one of many *slave threads*. For given data from the master thread, a slave thread decides whether or not there are fully normalized matrices in $\mathcal{D}(p)$ by calling $M(m, n, p - 1, p - 1)$ where (m, n) is the successor of (r, s) . Of course, in our parallel program, the master thread also has the role of *jobs scheduler*, i.e., the management of slave threads.

A choice of the *dividing index* (r, s) (i.e., the finishing index of the master thread) depends on the specific total order \preceq . We checked the three types of total orders, that is, \preceq_D , $\preceq_{D'}$ and \preceq_H . (See Example 2.4.1.) The figure 2 and 3 show respectively the cases of $p = 7$ and $p = 11$. The X-axis of the figures stands for choices of the dividing

(r, s)	$M(2, 2, r, s)$	#Partial results
(2, 2)	ϵ seconds	14
(2, 3)	ϵ seconds	157
(2, 4)	ϵ seconds	1507
(2, 5)	ϵ seconds	12327
(2, 6)	ϵ seconds	84573
(2, 7)	ϵ seconds	478501
(2, 8)	1 seconds	2186161
(2, 9)	1 seconds	7865605
(2, 10)	5 seconds	21644469
(2, 11)	12 seconds	43828409
(2, 12)	29 seconds	61675825

(r, s)	$M(2, 2, r, s)$	#Partial results
(2, 13)	50 seconds	55494757
(2, 14)	69 seconds	28008069
(2, 15)	81 seconds	6275119
(2, 16)	81 seconds	6275119
(3, 2)	85 seconds	37464544
(3, 3)	112 seconds	376242051
(3, 4)	335 seconds	2737088388
(3, 5)	1852 seconds	15753030361
(3, 6)	9878 seconds	71394611311
\vdots	\vdots	\vdots

FIGURE 4. The computation data in the case $p = 17$.

indices (r, s) among total orders, and the Y-axis means the corresponding counts of possibility for the partial results which is carried out by $M(2, 2, r, s)$. We see that the horizontal order is most efficient in the three types. Therefore we adopt the horizontal order in the case of $p = 17$ too, and in this case we choose the dividing index as $(2, 16)$ as Figure 4 suggested.

The specification of parallel computation for $p = 17$ is the following:

Fujitsu PRIMERGY CX400 [2];
CPU: Intel Xeon E5-2680 (2.7GHz, 8core) $\times 2$ / node;
Memory: 128GB / node
Interconnection network: InfiniBand FDR1 6.78GB/sec
Server system total peak performance: 811.86TFLOPS (1476 nodes)
OS: Red Hat Enterprise Linux;
Programming language: C with MPI (message passing interface);
Total number of processes: 1 (master) + 63 (slaves) = 64;
Total required time: 246093 seconds (\doteq 68 hours);

As mentioned in introduction, we obtain Theorem 1.1 as a result.

3. DESARGUESIAN PROJECTIVE PLANE YIELDS THE FOURIER MATRIX.

Let \mathcal{A} be a nonempty finite set and \mathcal{B} a family of subsets of \mathcal{A} . We say that $\rho \in \text{Sym}(\mathcal{A} \cup \mathcal{B})$ is an *automorphism* of $(\mathcal{A}, \mathcal{B})$ if, for all $(a, B) \in \mathcal{A} \times \mathcal{B}$, $a \in B$ if and only if $\rho(a) \in \rho(B)$. We denote by $\text{Aut}(\mathcal{A}, \mathcal{B})$ the group of automorphisms of $(\mathcal{A}, \mathcal{B})$.

For a positive integer $k \geq 2$ a pair $\mathcal{D} = (\mathcal{P}, \mathcal{L})$ is called a *projective plane of order k* if $|\mathcal{P}| = |\mathcal{L}| = k^2 + k + 1$, $|\{x \in \mathcal{P} \mid x \in L\}| = k + 1$ for each $L \in \mathcal{L}$ and $|\{x \in \mathcal{P} \mid x \in L \cap L'\}| = 1$ for all distinct $L, L' \in \mathcal{L}$. A pair $(x, L) \in \mathcal{P} \times \mathcal{L}$ is called a *flag* of \mathcal{D} if $x \in L$. For a flag (x, L) of \mathcal{D} we say that $\sigma \in \text{Aut}(\mathcal{P}, \mathcal{L})$ is an *elation* with respect to (x, L) if σ fixes each point in L and each line through x .

Let $\mathcal{D} = (\mathcal{P}, \mathcal{L})$ be a projective plane of order p containing an elation σ of order p with respect to a flag (x, L) . Let $y, z \in \mathcal{P} \setminus L$ be such that

x, y and z are not on a common line. For $i \in \{0, 1, \dots, p-1\}$ we define $N_i \in \mathcal{L}$ to be the line through y and $\sigma^i(z)$, and $y_i \in \mathcal{P}$ to be the point incident to N_0 and $\sigma^{-i}(N_1)$.

Lemma 3.1. *For all $i, j \in \{0, \dots, p-1\}$ there is a unique $E_{i,j} \in \mathbb{F}_p$ such that $\sigma^{E_{i,j}}(y_i) \in N_j$. Moreover $(E_{i,j})$ is fully normalized in $\mathcal{D}(p)$.*

Proof. Since $y_i \in \mathcal{P} \setminus L$ and $x \notin N_j$, the line M through x and y_i intersects N_j at exactly one point. Since σ acts regularly on $M \setminus \{x\}$, the first assertion follows. Since $y_i \in N_0$ and $y = y_0 \in N_i$ we have $E_{i,0} = E_{0,i} = 0$ for each $i \in \{0, 1, \dots, p-1\}$. Since $y_i \in \sigma^{-i}(N_1)$ and $\sigma^i(y_1) = \sigma^i(z) \in N_i$ we have $\sigma^i(y_i) \in N_1$ and $\sigma^i(y_1) \in N_i$ whence $E_{i,1} = E_{1,i} = i$ for each $i \in \{0, 1, \dots, p-1\}$. Suppose that $E_{i,k} - E_{j,k}$ ($k = 0, 1, \dots, p-1$) are not distinct for some $i \neq j$, i.e., $E_{i,k} - E_{j,k} = E_{i,l} - E_{j,l}$ for some $k \neq l$. Since $\sigma^{E_{i,k}}(y_i), \sigma^{E_{j,k}}(y_j) \in N_k$ and $\sigma^{E_{i,l}}(y_i), \sigma^{E_{j,l}}(y_j) \in N_l$ it follows that

$$\sigma^{E_{i,k}-E_{j,k}}(y_i), y_j \in \sigma^{-E_{j,k}}(N_k) \cap \sigma^{-E_{j,l}}(N_l).$$

Since $y_j \notin \langle \sigma \rangle y_i$ and $N_k \notin \langle \sigma \rangle N_l$ we have a contradiction. This completes the proof of the second assertion. \square

We fix the point set \mathcal{P} of size $p^2 + p + 1$. Let Δ be the set of all quadruples $(\mathcal{D}, \sigma, y, z)$ satisfying the following conditions:

- (i) $\mathcal{D} = (\mathcal{P}, \mathcal{L})$ is a projective plane of order p ;
- (ii) σ is an elation of \mathcal{D} with respect to a flag (x, L) ;
- (iii) $y, z \in \mathcal{P} \setminus L$ such that x, y, z are not on a common line.

By Lemma 3.1 we define a function Ψ from Δ to the set of all fully normalized Butson-Hadamard matrices of type (p, p) by $\Psi(\mathcal{D}, \sigma, x, y) = (\xi_p^{E_{i,j}})$ where $\sigma^{E_{i,j}}(y_i) \in N_j$.

Lemma 3.2. *The function Ψ is surjective.*

Proof. Let $H \in \text{BH}(p, p)$ be fully normalized. Then $H = (\xi_p^{E_{i,j}})$ where $(E_{i,j}) \in \mathcal{D}(p)$ is also fully normalized. Let C denote the $p \times p$ permutation matrix corresponding to the map from \mathbb{F}_p to itself defined by $\alpha \mapsto \alpha + 1$. We denote the $p^2 \times p^2$ matrix $(C^{E_{i,j}})$ by $P(H)$. We denote the $m \times n$ all one and zero matrix by $J_{m,n}$ and $O_{m,n}$ respectively, and we define $Q(H)$ to be a $(p^2 + p + 1) \times (p^2 + p + 1)$ matrix such that

$$Q(H) = \left(\begin{array}{c|c|c} 1 & J_{1,p} & O_{1,p^2} \\ \hline J_{p,1} & O_{p,p} & D \\ \hline O_{p^2,1} & D^T & P(H) \end{array} \right)$$

where D is a $p \times p^2$ matrix and

$$D = \left(\begin{array}{c|c|c|c} J_{1,p} & O_{1,p} & \cdots & O_{1,p} \\ \hline O_{1,p} & J_{1,p} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & O_{1,p} \\ \hline O_{1,p} & \cdots & O_{1,p} & J_{1,p} \end{array} \right).$$

Note that $Q(H)$ forms an incidence matrix of a projective plane of order p and

$$RQ(H)R^t = Q(H) \quad \text{where} \quad R = \left(\begin{array}{c|c} I_{p+1} & O_{p+1,p^2} \\ \hline O_{p^2,p+1} & I_p \otimes C \end{array} \right).$$

This implies that the projective plane \mathcal{D} having its incidence matrix $Q(H)$ has an elation σ with respect to the flag corresponding to the $(0,0)$ -entry of $Q(H)$. Let y, z be the points corresponding to the $(p+1)$ -th row and $(2p+1)$ -th row of $Q(H)$, respectively. Then the quadruple $(\mathcal{D}, \sigma, y, z)$ is mapped to H by Ψ . Therefore Ψ is surjective. \square

Lemma 3.3. *If \mathcal{D} is a Desarguesian projective plane of order p then $\Psi(\mathcal{D}, \sigma, y, z)$ is (ξ_p^{ij}) , namely, the Fourier matrix of degree p .*

Proof. Suppose $\mathcal{D} = (\mathcal{P}, \mathcal{L})$ is Desarguesian. Then the automorphism group of \mathcal{D} is isomorphic to $\text{PGL}(3, p)$. Let σ be an elation of order p with respect to a flag (x, L) and let $y, z \in \mathcal{L}$ be such that x, y, z are not in a common line. We denote by G the normalizer of $\langle \sigma \rangle$ in $\text{Aut}(\mathcal{P}, \mathcal{L})$. It is known that G acts doubly transitively on $\mathcal{P} \setminus L$ and $G \simeq \text{AGL}(2, p)$, and hence $G_{y,z} \simeq \text{AGL}(1, p)$ where we denote by $G_{y,z}$ the stabilizer subgroup fixing y and z . Note that $G_{y,z}$ contains τ which acts regularly on $\{y_i \mid i = 1, 2, \dots, p-1\}$ and regularly on $\{N_i \mid i = 1, 2, \dots, p-1\}$.

Suppose $\tau(N_1) = N_j$ for some j . Since $\sigma(y_1) \in N_1$ by the assumption and $y_1 = z$,

$$(\tau\sigma\tau^{-1})(y_1) \in \tau(N_1) = N_j.$$

Since $\sigma^j(y_1) \in N_j$ by the assumption and $\tau\sigma\tau^{-1}(y_1) \in N_j$ it follows that $\tau\sigma\tau^{-1} = \sigma^j$. Since $\tau(y_i) = y_i$ and $\sigma^i(y_i) \in N_1$ we have

$$\tau\sigma^i\tau^{-1}(y_i) \in \tau(N_1) = N_j.$$

On the other hand, since $\tau\sigma\tau^{-1} = \sigma^j$ we have $\tau\sigma^i\tau^{-1} = \sigma^{ij}$. Thus we have

$$\sigma^{ij}(y_i) = \tau\sigma^i\tau^{-1}(y_i) \in N_j.$$

This implies that we have $E_{i,j} = ij$ for all i and j . This completes the proof. \square

Proposition 3.4. *If there is a fully normalized matrix in $\text{BH}(p, p)$ which is not the Fourier matrix then it induces a non-Desarguesian projective plane of order p .*

Proof. This is due to the contrapositive of Lemma 3.3. \square

From Theorem 1.1 we have the following result:

Corollary 3.5. *For a prime $p \leq 17$, there is no non-Desarguesian projective plane of order p .* \square

ACKNOWLEDGEMENTS

1. The computation was mainly carried out using the computer facilities at Research Institute for Information Technology, Kyushu University [2].
2. We would like to thank Dong-been Kim, a student of the R&E (research and education) lecture given by the first author for his basic algorithm to search Butson Hadamard matrices.

REFERENCES

- [1] A.T. Butson, Generalized Hadamard matrices, *Proc. Amer. Math. Soc.* 13 1962 894–898.
- [2] <http://www.cc.kyushu-u.ac.jp/scp/>

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